FLATLAND the Movie
Hypercube Worksheet

Goal

The purpose of this worksheet is to spend time thinking a bit more deeply about spatial dimensions and understand how to extend our intuition when dealing with mathematical objects outside of our familiar three-dimensional world.

What are Dimensions?

In mathematics, an object that has only a length is a one-dimensional (1-D) object. An object that has a length and width is said to have two dimensions. The Flatland universe is a 2-D object. Things having a length, width, and depth (or height) are three-dimensional (3-D). In our everyday lives, it appears to us as if everything in the universe in which we live is three-dimensional. A convenient way to represent dimensions in mathematics is by using axes that are at right angles to each other.

From Cubes to Hypercubes: Reasoning by Analogy

Spherius uses analogy based on Pointland and Lineland to convey the idea of the third dimension to Arthur Square. The following exercises are analogous to what a fourth dimensional being might use to prove to us that a four-dimensional cube exists. We might at first be as skeptical about this analogy as Arthur Square was, but these exercises will help develop your intuition. Let us generally consider an “N-cube” as an N-dimensional object consisting of groups of opposite parallel line segments of equal length that are aligned in each of the space's N-dimensions at right angles to each other.
1. Sketch a 1-cube (line), 2-cube (square), and 3-cube.

<table>
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<th>1-cube</th>
<th>2-cube</th>
<th>3-cube</th>
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2. (a) What are the numbers of vertices (corners) for the N-cubes you drew in problem #1?

1-cube: 

2-cube: 

3-cube: 

(b) A cube in greater than three dimensions is called a hypercube. Since our brains are accustomed to 3-D, we cannot easily visualize hypercubes, just as Arthur Square could not visualize a 3-D sphere. What would you expect the number of vertices to be for a 4-D hypercube (4-cube)?

Answer:

(c) What is the general formula for the number of vertices of a hypercube in N dimensions?

Answer:
3.(a) What is the number of edges (lines) associated with a cube in dimensions 1, 2, and 3?

1-cube:

2-cube:

3-cube:

(b) What would you expect the number of edges to be for a 4-D hypercube? Hint: See if you can visually extend the pattern for the edges in the figures you drew in problem #1.

Answer:

(c) What is the general formula for the number of edges of a hypercube in N dimensions?

Answer:
4. Hex reasoned that a moving point traces out a line segment, and that a line segment moving parallel to itself makes a square. Continuing with her logic, Hex realized that a square moving parallel to itself would make a “super-square”, a 3-D cube. So, a cube in dimension N can be swept out by an N-1 dimensional cube (sometimes called an N-1 dimensional “face”) moving “parallel to itself”.

Let’s think about how many N-1 dimensional faces are on the boundary of an N-cube. For example, if we move a point (0-cube), we sweep out a line, which has 2 points on its boundary. If we move a line parallel to itself, we generate a square, which has 4 lines on its boundary (i.e., the square has four 1-D faces). A moving square can sweep out a 3-cube, which has 6 square 2-D faces. If a 3-D cube sweeps out a 4-D hypercube, how many 3-D faces are on the boundary of the 4-D hypercube?

5. When Spherius dips through Flatland, his cross section appears as a circle whose circumference is becoming larger, then shrinking, and finally disappearing. Describe the cross-sections of a 3-D cube as it passes through Flatland “edge-first”. What if the 3-D cube passes through Flatland “corner-first”?